

Generalized quandle polynomials

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Abstract

We define a family of generalizations of the two-variable quandle polynomial. These polynomial invariants generalize in a natural way to eight-variable polynomial invariants of finite biquandles. We use these polynomials to define a family of link invariants which further generalize the quandle counting invariant.

KEYWORDS: Finite quandles, finite biquandles, link invariants
2000 MSC: 57M27, 176D99

1 Introduction

In [11] a two-variable polynomial invariant of finite quandles, denoted $qp_Q(s, t)$, was introduced. This invariant was shown to distinguish all non-Latin quandles of order 5 and lower. A slight modification gives an invariant of subquandles embedded in larger quandles which is capable of distinguishing isomorphic subquandles embedded in different ways. This subquandle polynomial was used to augment the quandle counting invariant $|\text{Hom}(Q(L), T)|$ to obtain a multiset-valued invariant which can distinguish knots and links with the same quandle counting invariant value.

In this paper we generalize the quandle polynomial in two ways. In section 2 we define a family of two-variable polynomial invariants of finite quandles indexed by pairs of integers, denoted $qp_{m,n}(Q)$, such that $qp_Q(s, t) = qp_{1,1}(Q)$. In section 3 we extend these generalized quandle polynomials in a natural way to obtain a family of eight-variable polynomial invariants of finite biquandles indexed by pairs of integers, denoted $bp_{m,n}(B)$. In section 4 we define and give examples of link invariants defined using generalized quandle polynomials. In section 5 we collect a few questions for future research.

2 The (m, n) quandle polynomial

We begin with a definition from [7].

Definition 1 A *quandle* is a set Q with a binary operation $\triangleright : Q \times Q \rightarrow Q$ satisfying

- (i) for every $a \in Q$, $a \triangleright a = a$,
- (ii) for every $a, b \in Q$, there is a unique $c \in Q$ such that $a = c \triangleright b$, and
- (iii) for every $a, b, c \in Q$ we have $(a \triangleright b) \triangleright c = (a \triangleright c) \triangleright (b \triangleright c)$.

Axiom (ii) is equivalent to

- (ii') There is a second operation $\triangleright^{-1} : Q \times Q \rightarrow Q$ such that $(a \triangleright b) \triangleright^{-1} b = a = (a \triangleright^{-1} b) \triangleright b$.

As a useful abbreviation, let us denote the n -times repeated quandle operation as

$$x \triangleright^n y = (\dots (x \triangleright y) \triangleright y \dots) \triangleright y$$

and

$$x \triangleright^{-n} y = (\dots (x \triangleright^{-1} y) \triangleright^{-1} y \dots) \triangleright^{-1} y$$

where as expected, n is the number of triangles. Note that $x \triangleright^0 y = x$ for all x, y .

A standard example of a quandle is any group G , which has quandle structures given by $g \triangleright h = h^{-n}gh^n$ for $n \in \mathbb{Z}$ and $g \triangleright h = t(gh^{-1})h$ for any $t \in \text{Aut}(G)$, $g, h \in G$. The special case of the latter where G is abelian is called an *Alexander quandle* and may be regarded as a module over $\mathbb{Z}[t^{\pm 1}]$ by thinking of $t \in \text{Aut}(G)$ as a formal variable; in additive notation we have

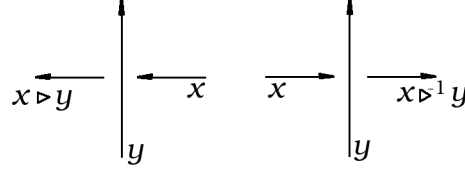
$$x \triangleright y = tx + (1 - t)y.$$

Another standard example of a quandle structure is any module V over a commutative ring R with an antisymmetric bilinear form¹ $\langle \cdot, \cdot \rangle : V \times V \rightarrow R$ with $\mathbf{x} \triangleright \mathbf{y} = \mathbf{x} + \langle \mathbf{x}, \mathbf{y} \rangle \mathbf{y}$. When R is a field and $\langle \cdot, \cdot \rangle$ is nondegenerate, V is a symplectic vector space, so this type of quandle is called a *symplectic quandle*. If $\langle \cdot, \cdot \rangle$ is instead a symmetric bilinear form, then the subset $S = \{\mathbf{x} \in V : \langle \mathbf{x}, \mathbf{x} \rangle \neq 0\} \subset V$ is a quandle under

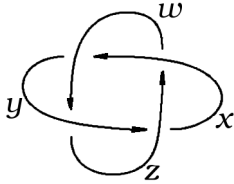
$$\mathbf{x} \triangleright \mathbf{y} = 2 \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\langle \mathbf{y}, \mathbf{y} \rangle} \mathbf{y} - \mathbf{x}$$

called a *Coxeter quandle*.

Another important example is the *knot quandle* $Q(L)$ defined in [7] which associates a quandle generator to every arc in a link diagram L and a relation at every crossing. The elements of a knot quandle are equivalence classes of quandle words in the generators modulo the equivalence relation generated by the quandle axioms and the relations imposed by the crossings.



Example 1 The two-component link pictured has knot quandle given by the listed presentation.



$$Q(L) = \langle x, y, z, w \mid x \triangleright z = y, z \triangleright y = w, y \triangleright w = x, w \triangleright x = z \rangle.$$

For symbolic computation, it is convenient to represent a finite quandle $Q = \{x_1, x_2, \dots, x_n\}$ with an $n \times n$ matrix M which encodes the operation table of Q , i.e., $M_{ij} = k$ where $x_k = x_i \triangleright x_j$. For example, the finite Alexander quandle $Q = \mathbb{Z}[t^{\pm 1}]/(2, t^2 + 1) = \{x_1 = 0, x_2 = 1, x_3 = t, x_4 = 1 + t\}$ has quandle operation matrix

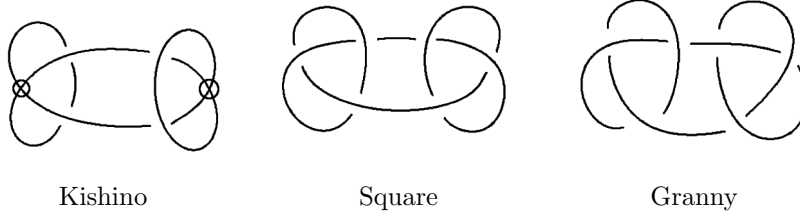
$$M_Q = \begin{bmatrix} 1 & 4 & 4 & 1 \\ 3 & 2 & 2 & 3 \\ 2 & 3 & 3 & 2 \\ 4 & 1 & 1 & 4 \end{bmatrix}.$$

Quandles have been much studied in recent years; see [7, 4, 10, 1, 15, 3], etc. Finite quandles are of particular interest since they can be used to define an easily computable invariant of knots and links,

¹If the characteristic of R is 2, then we require that $\langle \mathbf{x}, \mathbf{x} \rangle = 0$.

the *quandle counting invariant* $|\text{Hom}(Q(L), T)|$. Each quandle homomorphism $f \in \text{Hom}(Q(L), T)$ can be pictured as a coloring of a link diagram representing the link L with a quandle element $f(x) \in T$ attached to each arc x satisfying the crossing condition pictured above.

It is well known that quandle and biquandle counting invariants are stronger than fundamental group counting invariants (i.e., counting homomorphisms from a knot group to a finite group). For example, the *Kishino virtual knot* below is distinguished from the unknot by biquandle counting invariants despite having trivial knot group; see [13]. Similarly, the knot quandles of the square knot SK and granny knot GK are known to be nonisomorphic since their generalized knot groups G_n derived from their knot quandles are distinguished by counting invariants for $n \geq 2$ in [14]. We suspect that the corresponding conjugation quandle counting invariants should distinguish the knots, though direct computational confirmation of this is difficult due to the size of the groups involved. However, the fact that $\pi_1(S^3 \setminus GK) \cong \pi_1(S^3 \setminus SK)$ implies that no group counting invariant $|\text{Hom}(G, T)|$ where G is the usual knot group can distinguish the square knot from the granny knot.



In [11] we have the following definition:

Definition 2 Let Q be a finite quandle. For any element $x \in Q$, let

$$C(x) = \{y \in Q : y \triangleright x = y\} \quad \text{and} \quad R(x) = \{y \in Q : x \triangleright y = x\}$$

and set $r(x) = |R(x)|$ and $c(x) = |C(x)|$. Then the *quandle polynomial* of Q , $qp_Q(s, t)$, is

$$qp_Q(s, t) = \sum_{x \in Q} s^{r(x)} t^{c(x)}.$$

An isomorphism $\phi : Q \rightarrow Q'$ induces bijections $\phi_r : R(x) \rightarrow R(\phi(x))$ and $\phi_c : C(x) \rightarrow C(\phi(x))$, so $qp_Q(Q) = qp_{Q'}(Q')$ and qp_Q is an invariant of isomorphism type for finite quandles.

We can now make our first new definition.

Definition 3 Let Q be a finite quandle. For any element $x \in Q$, let

$$C_n(x) = \{y \in Q : y \triangleright^n x = y\} \quad \text{and let} \quad R_m(x) = \{y \in Q : x \triangleright^m y = x\}$$

and set $r_m(x) = |R_m(x)|$ and $c_n(x) = |C_n(x)|$. Then the (m, n) -quandle polynomial of Q , $qp_{m,n}(Q)$, is

$$qp_{m,n}(Q) = \sum_{x \in Q} s^{r_m(x)} t^{c_n(x)}.$$

An isomorphism $\phi : Q \rightarrow Q'$ then induces for each $x \in Q$ bijections $\phi_{r,m} : R_m(x) \rightarrow R_m(\phi(x))$ and $\phi_{c,n} : C_n(x) \rightarrow C_n(\phi(x))$. It follows that $qp_{m,n}(Q) = qp_{m,n}(Q')$ and qp_Q is an invariant of quandle isomorphism for finite quandles.

Example 2 From the definition, we have $qp_{0,0}(Q) = |Q|s^{|Q|}t^{|Q|}$ and $qp_{1,1}(Q) = qp_Q(Q)$.

For any element $y \in Q$, the second quandle axiom implies that y acts on Q by a permutation $\rho_y \in S_{|Q|}$ (where $S_{|Q|}$ is the symmetric group on $|Q|$ letters) given by the column corresponding to y in the matrix of Q . Let $\text{ord}(\rho)$ denote the order of ρ in $S_{|Q|}$, i.e., the cardinality of the cyclic subgroup of $S_{|Q|}$ generated by ρ . Then if $n \equiv n' \pmod{\text{ord}(\rho_y)}$, we have $x \triangleright^n y = x \triangleright^{n'} y$. It follows that for any finite quandle Q , there are at most N^2 distinct generalized quandle polynomials where $N = \text{lcm}\{\text{ord}(\rho_y) : y \in Q\}$. In particular, to find all generalized quandle polynomials it suffices to consider the subset $\{qp_{m,n} \mid 0 \leq m, n \leq N-1\}$ of the \mathbb{Z}^2 -lattice of generalized quandle polynomials. For ease of comparison, we can write these entries in an $N \times N$ matrix $M_{qp}(Q)$ whose (i, j) entry is $qp_{i-1, j-1}(Q)$, which we will call the *generalized quandle polynomial matrix* of Q . Both the size and the entries of this matrix are invariants of quandle isomorphism.

Example 3 The Alexander quandle $Q = \mathbb{Z}[t^{\pm 1}]/(2, t^2 + 1) = \{x_1 = 0, x_2 = 1, x_3 = t, x_4 = 1 + t\}$ has $\text{ord}(\rho_y) = 2$ for all $y \in Q$, so $N = 2$; we compute the generalized quandle polynomials $qp_{0,0} = 4s^4t^4$, $qp_{0,1} = 4s^2t^4$, $qp_{1,0} = 4s^4t^2$, and $qp_{1,1} = 4s^2t^2$. Thus, Q has generalized quandle polynomial matrix

$$M_{qp}(Q) = \begin{bmatrix} 4s^4t^4 & 4s^2t^4 \\ 4s^4t^2 & 4s^2t^2 \end{bmatrix}.$$

Example 4 A quandle is *strongly connected* or *Latin* if its operation matrix is a Latin square, that is, if its rows as well as columns are permutations (see [6]). In [11], **Maple** computations showed that $qp_{1,1}(Q)$ distinguishes all non-Latin quandles of cardinality up to 5, while Latin quandles always have $qp_{1,1}(Q) = |Q|st$. Our **Maple** computations show that of the three Latin quandles with five elements, two have the same generalized quandle polynomial matrix while one has a different matrix. This shows that the generalized quandle polynomials contain additional information about quandle isomorphism type not contained in $qp_Q(s, t)$, while still not determining the quandle's isomorphism type for Latin quandles. See Table 1.

Quandle matrix	Generalized quandle polynomial matrix
$\begin{bmatrix} 1 & 5 & 4 & 3 & 2 \\ 3 & 2 & 1 & 5 & 4 \\ 5 & 4 & 3 & 2 & 1 \\ 2 & 1 & 5 & 4 & 3 \\ 4 & 3 & 2 & 1 & 5 \end{bmatrix}$	$\begin{bmatrix} 5s^5t^5 & 5st^5 & 5st^5 & 5st^5 \\ 5s^5t & 5st & 5st & 5st \\ 5s^5t & 5st & 5st & 5st \\ 5s^5t & 5st & 5st & 5st \end{bmatrix}$
$\begin{bmatrix} 1 & 4 & 2 & 5 & 3 \\ 4 & 2 & 5 & 3 & 1 \\ 2 & 5 & 3 & 1 & 4 \\ 5 & 3 & 1 & 4 & 2 \\ 3 & 1 & 4 & 2 & 5 \end{bmatrix}$	$\begin{bmatrix} 5s^5t^5 & 5st^5 & 5st^5 & 5st^5 \\ 5s^5t & 5st & 5st & 5st \\ 5s^5t & 5st & 5st & 5st \\ 5s^5t & 5st & 5st & 5st \end{bmatrix}$
$\begin{bmatrix} 1 & 3 & 5 & 2 & 4 \\ 5 & 2 & 4 & 1 & 3 \\ 4 & 1 & 3 & 5 & 2 \\ 3 & 5 & 2 & 4 & 1 \\ 2 & 4 & 1 & 3 & 5 \end{bmatrix}$	$\begin{bmatrix} 5s^5t^5 & 5st^5 \\ 5s^5t & 5st \end{bmatrix}$

Table 1: Generalized quandle polynomial matrices of Latin quandles of cardinality 5.

3 Biquandle polynomials

In the section we define the analog of generalized quandle polynomials for finite biquandles. We start with the definition of a biquandle, also known as a type of *switch* or *Yang-Baxter Set*; see [8].

Definition 4 A *biquandle* is a set B with four binary operations $B \times B \rightarrow B$ denoted by

$$(a, b) \mapsto a^b, a^{\bar{b}}, a_b, \text{ and } a_{\bar{b}}$$

respectively, satisfying the following axioms:

1. For every pair of elements $a, b \in B$, we have

$$(i) a = a^{b\bar{b}_a}, \quad (ii) b = b_{a\bar{a}^b}, \quad (iii) a = a^{\bar{b}b_a}, \quad \text{and} \quad (iv) b = b_{\bar{a}a^b}.$$

2. Given elements $a, b \in B$, there are elements $x, y \in B$ such that

$$(i) x = a^{b_x}, \quad (ii) a = x^{\bar{b}}, \quad (iii) b = b_{\bar{x}a},$$

$$(iv) y = a^{\bar{b}_y}, \quad (v) a = y^b, \quad \text{and} \quad (vi) b = b_{y\bar{a}}.$$

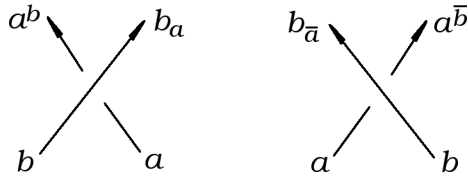
3. For every triple $a, b, c \in B$ we have:

$$(i) a^{bc} = a^{c_b b^c}, \quad (ii) c_{ba} = c_{a^b b_a}, \quad (iii) (b_a)^{c_a b} = (b^c)_{a^{c_b}},$$

$$(iv) a^{\bar{b}\bar{c}} = a^{\bar{c}_{\bar{b}} \bar{b}^{\bar{c}}}, \quad (v) c_{\bar{b}\bar{a}} = c_{\bar{a}^{\bar{b}} \bar{b}_{\bar{a}}}, \quad \text{and} \quad (vi) (b_{\bar{a}})^{\bar{c}_{\bar{a}} \bar{b}} = (b^{\bar{c}})_{\bar{a}^{\bar{c}_{\bar{b}}}}.$$

4. Given an element $a \in B$, there are *unique* elements $x, y \in B$ such that

$$(i) x = a_x, \quad (ii) a = x^a, \quad (iii) y = a^{\bar{y}}, \quad \text{and} \quad (iv) a = y_{\bar{a}}.$$



The biquandle axioms come from dividing an oriented link diagram into semi-arcs at every over and under crossing point; then both inbound semiarcs act on each other at both positive and negative crossings, for a total of four binary operations $(a, b) \mapsto a^b, a^{\bar{b}}, a_b$ and $a_{\bar{b}}$. The axioms are then the result of transcribing a minimal set of oriented Reidemeister moves. See [8] for more.

One standard example of a biquandle is any quandle, which is a biquandle under

$$a^b = a \triangleright b, \quad a^{\bar{b}} = a \triangleright^{-1} b, \quad a_b = a_{\bar{b}} = a$$

as well as under

$$a^b = a \triangleright^{-1} b, \quad a^{\bar{b}} = a \triangleright b, \quad a_b = a_{\bar{b}} = a,$$

$$a_b = a \triangleright b, \quad a_{\bar{b}} = a \triangleright^{-1} b, \quad a^b = a^{\bar{b}} = a$$

and

$$a_b = a \triangleright^{-1} b, \quad a_{\bar{b}} = a \triangleright b, \quad a^b = a^{\bar{b}} = a.$$

Another standard example of a biquandle structure is any module over $\mathbb{Z}[t^{\pm 1}, s^{\pm 1}]$ with

$$a^b = ta + (1 - st)b, \quad a^{\bar{b}} = t^{-1}a + (1 - s^{-1}t^{-1})b, \quad a_b = sa, \quad \text{and} \quad a_{\bar{b}} = s^{-1}b;$$

biquandles of this type are called *Alexander biquandles*. For a concrete example, take $B = \mathbb{Z}_n$ and let $s, t \in B$ be any two invertible elements. See [8] and [9].

Much like with finite quandles, we can represent a finite biquandle $B = \{x_1, \dots, x_n\}$ with a block matrix encoding the operation tables of the four operations:

$$M_B = \left[\begin{array}{c|c} B^1 & B^2 \\ \hline B^3 & B^4 \end{array} \right] \quad B_{ij}^k = m \quad \text{where} \quad x_m = \begin{cases} (x_i)^{\overline{(x_j)}} & k = 1 \\ (x_i)^{(x_j)} & k = 2 \\ (x_i)_{\overline{(x_j)}} & k = 3 \\ (x_i)_{(x_j)} & k = 4. \end{cases}$$

Example 5 The Alexander biquandle $B = \mathbb{Z}_4$ with $s = 3$ and $t = 1$ has biquandle matrix

$$M_B = \left[\begin{array}{cccc|cccc} 3 & 1 & 3 & 1 & 3 & 1 & 3 & 1 \\ 4 & 2 & 4 & 2 & 4 & 2 & 4 & 2 \\ 1 & 3 & 1 & 3 & 1 & 3 & 1 & 3 \\ 2 & 4 & 2 & 4 & 2 & 4 & 2 & 4 \\ \hline 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \end{array} \right] \quad \text{where} \quad B = \{x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 0\}.$$

In what follows, we will find it convenient to use the notation

$$\text{op}_1(x, y) = x^{\bar{y}}, \quad \text{op}_2(x, y) = x^y, \quad \text{op}_3(x, y) = x_{\bar{y}} \quad \text{and} \quad \text{op}_4(x, y) = x_y,$$

and as before $\text{op}_i^n(x, y) = \text{op}_i(\dots \text{op}_i(\text{op}_i(x, y), y) \dots y)$, where n is the number of “ op_i ”s.

We can extend the generalized quandle polynomials in the obvious way to obtain an invariant of biquandles up to isomorphism:

Definition 5 Let B be a finite biquandle. For every $x \in B$, define

$$C_n^i(x) = \{y \in B \mid \text{op}_i^n(y, x) = y\} \quad \text{and} \quad R_m^i(x) = \{y \in B \mid \text{op}_i^m(x, y) = x\}$$

where $m, n \in \mathbb{Z}$. Let $c_n^i(x) = |C_n^i(x)|$ and $r_m^i(x) = |R_m^i(x)|$ for $i = 1, \dots, 4$. Then the (m, n) *biquandle polynomial* of B is

$$bp_{m,n}(B) = \sum_{x \in B} s_1^{r_m^1(x)} s_2^{r_m^2(x)} s_3^{r_m^3(x)} s_4^{r_m^4(x)} t_1^{c_n^1(x)} t_2^{c_n^2(x)} t_3^{c_n^3(x)} t_4^{c_n^4(x)}.$$

Example 6 As in the quandle case, we have

$$bp_{0,0}(B) = |B| s_1^{|B|} t_1^{|B|} s_2^{|B|} t_2^{|B|} s_3^{|B|} t_3^{|B|} s_4^{|B|} t_4^{|B|}$$

for every finite biquandle. Also as in the quandle case, specializing $s_i = t_i = 1$ for all $i = 1, \dots, 4$ yields $|B|$ for all $m, n \in \mathbb{Z}$.

Example 7 Every quandle Q is a biquandle with $a^b = a \triangleright b$, $a^{\bar{b}} = a \triangleright^{-1} b$ and $a_b = a_{\bar{b}} = a$. Specializing $s_2 = s$, $t_2 = t$ and $s_i = t_i = 1$ for $i = 1, 3, 4$ in $bp_{m,n}(Q)$ yields $qp_{m,n}(Q)$.

Example 8 The Alexander biquandle $B = \mathbb{Z}_3$ with $s = 2, t = 1$ has biquandle matrix

$$M_B = \left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 3 & 2 & 1 \\ 1 & 3 & 2 & 1 & 3 & 2 \\ 2 & 1 & 3 & 2 & 1 & 3 \\ \hline 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 & 3 \end{array} \right].$$

We can compute $bp_{1,1}(B)$ by counting the number of times a row number appears in each column and row in each of the operation block matrices. Here we see that $bp_{1,1}(B) = 2s_1s_2t_3t_4 + s_1t_1^3s_2t_2^3s_3^3t_3s_4^3t_4$.

As before, a biquandle isomorphism $\phi : B \rightarrow B'$ induces bijections $\phi_{i,n} : C_n^i(x) \rightarrow C_n^i(\phi(x))$ and $\psi_{i,m} : R_m^i(x) \rightarrow R_m^i(\phi(x))$ for each $i = 1, \dots, 4$, and $n, m \in \mathbb{Z}$, so $B \cong B'$ implies $bp_{m,n}(B) = bp_{m,n}(B')$ and each $bp_{m,n}(B)$ is an invariant of biquandle isomorphism.

The columns in a finite biquandle matrix, like those in a quandle matrix, are permutations of $\{1, 2, \dots, |B|\}$. Let $\rho_1(y) \in S_{|B|}$, $\rho_2(y) \in S_{|B|}$, $\rho_3(y) \in S_{|B|}$, and $\rho_4(y) \in S_{|B|}$ be the permutations corresponding to the actions of y on B given by $\text{op}_i(_, y) : B \rightarrow B$. Then as in the quandle case, the fact that $n \equiv n' \pmod{\text{ord}(\rho_i(y))}$ implies $\text{op}_i^n(x, y) = \text{op}_i^{n'}(x, y)$ then implies that $bp_{m,n}(B) = bp_{m',n'}(B)$ if $n \equiv n' \pmod{N}$ and $m \equiv m' \pmod{N}$ where $N = \text{lcm}\{\text{ord}(\rho_i(y)) : y \in B, i = 1, \dots, 4\}$. Hence, as before, there are at most N^2 distinct biquandle polynomials for a biquandle B .

Thus, for every finite biquandle B , the *biquandle polynomial matrix* of B is the $N \times N$ matrix whose m, n entry is $bp_{m,n}(B)$. Continuing with example 7, if a biquandle B is a quandle with $a \triangleright b = a^b, a \triangleright^{-1} b = a^{\bar{b}}$ and $a_b = a_{\bar{b}} = a$, then specializing $s_1 = t_1 = s_3 = t_3 = s_4 = t_4 = 1$ and $s_2 = s, t_2 = t$ in the biquandle polynomial matrix of B yields the generalized quandle polynomial matrix $M_{qp}(B)$.

Example 9 The Alexander biquandle in example 5 has biquandle polynomial matrix

$$\begin{bmatrix} 4s_1^4t_1^4s_2^4t_2^4s_3^4t_3^4s_4^4t_4^4 & 2s_1^2t_1^4s_2^2t_2^4t_3^4t_4^4 + 2s_1^2t_1^4s_2^2t_2^4s_3^4t_3^4s_4^4t_4^4 \\ 2s_1^4s_2^4s_3^4t_3^4s_4^4t_4^2 + 2s_1^4t_1^4s_2^4t_2^4s_3^4t_3^4s_4^4t_4^2 & 2s_1^2s_2^2t_3^2t_4^2 + 2s_1^2t_1^4s_2^2t_2^4s_3^4t_3^4s_4^4t_4^2 \end{bmatrix}$$

according to our **Maple** computations.

Maple code for computing quandle and biquandle polynomials is available for download at www.esotericka.org/quandles. Computations with this code reveal that all isomorphism classes of biquandles with up to four elements are distinguished by $bp_{1,1}(B)$ alone, using the list of biquandle isomorphism classes from [13].²

4 Link invariants from generalized quandle polynomials

In this section we extend the subquandle polynomial defined in [11] to the $qp_{m,n}(Q)$ and $bp_{m,n}(B)$ settings and exhibit some examples of the resulting link invariants.

Definition 6 Let Q be a finite quandle and $S \subset Q$ a subquandle. Then for any $m, n \in \mathbb{Z}$, the *generalized subquandle polynomial* is

$$sqp_{m,n}(S \subset Q) = \sum_{x \in S} s^{r_m(x)} t^{c_n(x)}.$$

²Note that the published list contains a few small typographical errors, but one can regenerate the correct list with the **Maple** code.

Similarly, for any finite biquandle B with subbiquandle $S \subset B$ and $m, n \in \mathbb{Z}$ the *subbiquandle polynomial* is

$$sbp_{m,n}(S \subset B) = \sum_{x \in S} s_1^{r_m^1(x)} s_2^{r_m^2(x)} s_3^{r_m^3(x)} s_4^{r_m^4(x)} t_1^{c_n^1(x)} t_2^{c_n^2(x)} t_3^{c_n^3(x)} t_4^{c_n^4(x)}.$$

Thus, the subquandle and subbiquandle polynomials are the contributions to the quandle and biquandle polynomials coming from the subquandle or subbiquandle in question. The $(1, 1)$ subquandle polynomial was shown in [11] to encode information about how the subquandle S is embedded in Q ; indeed, $sqp_{1,1}$ can distinguish isomorphic subquandles embedded in different ways.

Since the image of a homomorphism of a knot quandle into a target quandle T is a subquandle of T , we can modify the quandle counting invariant to obtain a multiset-valued link invariant by counting the subquandle polynomial of $\text{Im}(f)$ for each $f \in \text{Hom}(Q(L), T)$. The cardinality of this multiset is then the usual counting invariant. This generalizes the specialized subquandle polynomial invariant which was shown in [11] to distinguish some links which have the same quandle counting invariant.

Definition 7 Let L be a link, T a finite quandle and $m, n \in \mathbb{Z}$. Then the multiset

$$\Phi_{sqp_{m,n}}(L, T) = \{sqp_{m,n}(\text{Im}(f) \subset T) : f \in \text{Hom}(Q(L), T)\}$$

is the (m, n) -subquandle polynomial invariant of L with respect to T . We can rewrite the multiset in a polynomial-style form by converting the multiset elements to exponents of a dummy variable q and converting their multiplicities to coefficients:

$$\phi_{sqp_{m,n}}(L, T) = \sum_{f \in \text{Hom}(Q(L), T)} q^{sqp_{m,n}(\text{Im}(f) \subset T)}.$$

If T is a finite biquandle, we similarly define the (m, n) -subbiquandle polynomial invariant to be the multiset

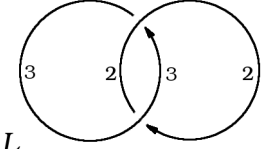
$$\Phi_{sbp_{m,n}}(L, T) = \{sbp_{m,n}(\text{Im}(f) \subset T) : f \in \text{Hom}(B(L), T)\}$$

or in polynomial form

$$\phi_{sbp_{m,n}}(L, T) = \sum_{f \in \text{Hom}(B(L), T)} q^{sbp_{m,n}(\text{Im}(f) \subset T)}.$$

Collecting all the of the subquandle or subbiquandle polynomial invariants into an $N \times N$ matrix whose (m, n) entry is $sqp_{m,n}(\text{Im}(f) \subset T)$ or $sbp_{m,n}(\text{Im}(f) \subset T)$ yields an invariant of links which includes information from all of the subquandle or subbiquandle polynomials. Specializing $s_i = t_i = 0$ for $i = 1, \dots, 4$ or specializing $q = 1$ in any entry of the matrix yields the appropriate counting invariant.

Example 10



L

$M_T = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 2 & 2 & 3 & 2 & 2 \\ 2 & 3 & 3 & 2 & 3 & 3 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 2 & 2 & 3 & 2 & 2 \\ 2 & 3 & 3 & 2 & 3 & 3 \end{array} \right]$

The Hopf link L has nine biquandle colorings by the three-element biquandle T below. In particular, the pictured coloring has image subbiquandle $\text{Im}(f) = \{2, 3\} \subset T$. The element $2 \in T$ has $r_i^i = 2$ and $c_i^i = 3$ for $i = 1, 2, 3, 4$, so $2 \in \text{Im}(f)$ contributes $s_1^2 s_2^2 s_3^2 s_4^2 t_1^3 t_2^3 t_3^3 t_4^3$ to the exponent of q

for this homomorphism. Indeed, the contribution from $3 \in T$ is the same, so we have a contribution of $q^{2s_1^2 s_2^2 s_3^2 s_4^2 t_1^3 t_2^3 t_3^3 t_4^3}$ from this homomorphism. Repeating with the other homomorphisms, we have

$$\begin{aligned}\phi_{sbp_{1,1}}(L, T) &= q^{s_1^3 t_1 s_2^3 t_2 s_3^3 t_3 s_4^3 t_4} + 4q^{s_1^3 t_1 s_2^3 t_2 s_3^3 t_3 s_4^3 t_4 + 2s_1^2 t_1^3 s_2^2 t_2^3 s_3^2 t_3^3 s_4^2 t_4^3} \\ &\quad + 2q^{s_1^2 t_1^3 s_2^2 t_2^3 s_3^2 t_3^3 s_4^2 t_4^3} + 2q^{2s_1^2 t_1^3 s_2^2 t_2^3 s_3^2 t_3^3 s_4^2 t_4^3}.\end{aligned}$$

Since the least common multiple of the orders of the columns of T is 2, the full invariant is a 2×2 matrix, of which the above value is one entry.

If K is a single-component link, that is, a knot, then the knot quandle of K is connected, and hence the image of any quandle homomorphism $f : Q(K) \rightarrow T$ must lie inside a single orbit subquandle of T . In particular, we have

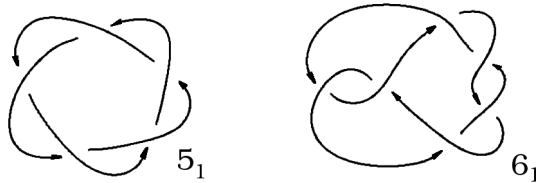
$$|\text{Hom}(Q(K), T)| = \left| \text{Hom} \left(Q(K), \bigcup_{i=1}^n T_i \right) \right| = \sum_{i=1}^n |\text{Hom}(Q(K), T_i)|$$

where T_i are the orbit subquandles of T . In practice, this has meant that multi-orbit quandles have been largely ignored in favor of single-orbit (“connected”) quandles. The next example demonstrates that by using generalized subquandle polynomials, non-connected quandles can still be used to distinguish knots whose counting invariants are the same.

Example 11 The quandle T' with operation matrix

$$M_{T'} = \begin{bmatrix} 1 & 3 & 5 & 2 & 4 & 3 & 1 & 4 & 2 & 5 \\ 5 & 2 & 4 & 1 & 3 & 5 & 3 & 1 & 4 & 2 \\ 4 & 1 & 3 & 5 & 2 & 2 & 5 & 3 & 1 & 4 \\ 3 & 5 & 2 & 4 & 1 & 4 & 2 & 5 & 3 & 1 \\ 2 & 4 & 1 & 3 & 5 & 1 & 4 & 2 & 5 & 3 \\ 8 & 9 & 10 & 6 & 7 & 6 & 10 & 9 & 8 & 7 \\ 7 & 8 & 9 & 10 & 6 & 8 & 7 & 6 & 10 & 9 \\ 6 & 7 & 8 & 9 & 10 & 10 & 9 & 8 & 7 & 6 \\ 10 & 6 & 7 & 8 & 9 & 7 & 6 & 10 & 9 & 8 \\ 9 & 10 & 6 & 7 & 8 & 9 & 8 & 7 & 6 & 10 \end{bmatrix}$$

has two 5-element orbit subquandles. The two knots 5_1 and 6_1 pictured below both have counting invariant $|\text{Hom}(Q(5_1), T')| = 30 = |\text{Hom}(Q(6_1), T')|$ with respect to T' . Indeed, the two knots have the same $\phi_{sqp_{1,1}}(K, T')$ value. However, the generalized subquandle polynomial invariants with $m = 2$ distinguish the knots, detecting the fact that the sets of homomorphisms are different despite having the same cardinality.



n	$\phi_{sqp_{2,n}}(5_1, T')$	$\phi_{sqp_{2,n}}(6_1, T')$
0	$5q^{s^{10}t^{10}} + 20q^{5s^{10}t^{10}} + 5q^{s^{10}t^2}$	$5q^{s^{10}t^2} + 20q^{5s^{10}t^2} + 5q^{s^{10}t^{10}}$
1	$5q^{s^2t^{10}} + 20q^{5s^2t^{10}} + 5q^{s^2t^2}$	$5q^{s^2t^2} + 20q^{5s^2t^2} + 5q^{s^2t^{10}}$
2	$5q^{s^6t^{10}} + 20q^{5s^6t^{10}} + 5q^{s^6t^2}$	$5q^{s^6t^2} + 20q^{5s^6t^2} + 5q^{s^6t^{10}}$
3	$5q^{s^2t^{10}} + 20q^{5s^2t^{10}} + 5q^{s^2t^2}$	$5q^{s^2t^2} + 20q^{5s^2t^2} + 5q^{s^2t^{10}}$

5 Questions for future research

The quandle and biquandle polynomials as currently defined only make sense for finite quandles. Consequentially, to use these polynomials for defining invariants of knot and link quandles and biquandles, which are typically infinite, we must first convert to finite quandles in some way. If a version of the quandle polynomial could be defined for arbitrary quandles, or perhaps just for finitely generated quandles such as knot quandles, we might use such a polynomial (or series?) to obtain link invariants more directly.

The only example so far of two non-isomorphic quandles or biquandles with the same (bi)quandle polynomial matrix is the pair of Latin quandles of order 5 from table 1. Are there examples of non-isomorphic non-Latin quandles or non-quandle biquandles which are not distinguished by their polynomial matrices?

If two knots or links have distinct quandles or biquandles, let the *sub(bi)quandle polynomial matrix index* of the pair be the cardinality of the smallest finite (bi)quandle whose polynomial matrix invariant distinguishes the pair, or ∞ if there is no such finite (bi)quandle. Is there a pair of knots or links whose sub(bi)quandle polynomial matrix index is infinite?

Each of the entries in a quandle or biquandle polynomial matrix has total coefficient equal to the cardinality of the (bi)quandle. What other relationships, if any, exist among the entries in a (bi)quandle polynomial matrix? In particular, what is the minimal subset of the entries which determines the other entries?

6 Acknowledgements

The author would like to thank Jozef Przytycki for his comments and observations at the Knots in Washington conference which directly inspired definition 3, as well as Thao-Nhi Luu, whose conversations with the author also influenced the direction of this paper, and the referee, whose comments and observations improved the paper.

References

- [1] J. S. Carter, M. Elhamdadi, M. Graña, and M. Saito. *Cocycle knot invariants from quandle modules and generalized quandle homology*. Osaka J. Math. **42** (2005) 499-541.
- [2] J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford and M. Saito. *Quandle cohomology and state-sum invariants of knotted curves and surfaces*. Trans. Amer. Math. Soc. **355** (2003) 3947-3989.
- [3] M. Eisermann. *Quandle coverings and their Galois correspondence*. arXiv:math/0612459v3
- [4] R. Fenn and C. Rourke. *Racks and links in codimension two*. J. Knot Theory Ramifications **1** (1992), 343-406.
- [5] R. Henderson, T. Macedo and S. Nelson. *Symbolic computation with finite quandles*. J. Symbolic Comput. **41** (2006) 811-817.
- [6] B. Ho and S. Nelson. *Matrices and finite quandles*. Homology Homotopy Appl. **7** (2005) 197-208.
- [7] D. Joyce. *A classifying invariant of knots, the knot quandle*. J. Pure Appl. Algebra **23** (1982) 37-65.
- [8] L. H. Kauffman and D. Radford. *Bi-oriented quantum algebras, and a generalized Alexander polynomial for virtual links*. Contemp. Math. **318** (2003) 113-140.

- [9] D. Lam and S. Nelson. *An isomorphism theorem for Alexander biquandles*. arXiv:math/0611887, to appear in Int'l. J. Math.
- [10] S. V. Matveev. *Distributive groupoids in knot theory*. Math. USSR, Sb. **47** (1984) 73-83.
- [11] S. Nelson. *A polynomial invariant of finite quandles*. arxiv:math.QA/0702038, to appear in J. Alg. Appl.
- [12] S. Nelson and W. Neumann. *The 2-generalized knot group determines the knot*. arXiv:0804.0807; to appear in Comm. Contemp. Math
- [13] S. Nelson and J. Vo. *Matrices and finite biquandles*. Homology, Homotopy and Applications **8** (2006) 51-73.
- [14] C. Tuffley. *Generalised knot groups distinguish the square and granny knots*. arXiv:0706.1807
- [15] J. Zablow. *Intersections of curves on surfaces with disk families in handlebodies*. J. Knot Theory Ramifications **15** (2006) 631-649.

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